AAA4090

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 926

THE INFLUENCE OF THE AERODYNAMIC SPAN EFFECT ON THE MAGNITUDE OF THE TORSIONAL-DIVERGENCE VELOCITY AND ON THE SHAPE OF THE CORRESPONDING DEFLECTION MODE

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SUMMARY

A procedure which takes into account the aerodynamic span effect is given for the determination of the torsional-divergence velocities of monoplanes.

The explicit solutions obtained in several cases indicate that the aerodynamic span effect may increase the divergence velocities found by means of the section-force theory by as much as 17 to 40 percent.

It is found that the magnitude of the effect increases with increasing degree of stiffness taper and decreases with increasing degree of chord taper.

By a slight extension of the present method it is possible to analyze the elastic deformations of wings, and the resultant lift distributions, before torsional divergence occurs.

INTRODUCTION

This paper deals with the limiting case of the bending-torsion flutter problem which occurs when the flutter frequency has the value zero. This aspect of the problem has been formulated and dealt with, as a problem of static torsional instability, by H. Reissner (reference 1) in 1926. Reissner's treatment, as well as the later work on the general flutter problem by Theodorsen (reference 2), Loring (reference 3), and Blakney (reference 4), assumes expressions for the relevant air forces at each section of the wing which correspond to the assumption of two-dimensional flow.

The purpose of the present paper is to investigate the effect of this simplifying assumption by giving a procedure for the analysis of the torsional-divergence problem which takes into account the aerodynamic span effect. The developments are based on the theory of torsion of straight rods and on lifting-line theory for the spanwise distribution of lift.

A rapidly convergent process of iteration is devised for the solution of the equations of the two theories for an elastically twisted wing. The method is applied to some typical examples and it is found that for a wing with an aspect ratio of about six the aerodynamic span effect modifies the torsional-divergence velocity obtained with the assumption of air forces of the two-dimensional theory by 17 to 40 percent, depending on the elastic and plan-form characteristics of the wing.

This investigation, conducted at the Massachusetts Institute of Technology, was sponsored by, and conducted with financial assistance from, the National Advisory Committee for Aeronautics.

SYMBOLS

b wing span

c wing chord

c_R root chord $(c^* = c/c_R)$

α_o angle of attack before elastic deformation

ϑ angle of twist due to elastic deformation

l section lift (per unit span)

ρ density of air

V velocity of flight

m profile constant $(dc_l/d\alpha_o)$

c_l section lift coefficient $\left(l/\frac{1}{2}\rho V^2 c\right)$

F auxiliary lift function $\left(cc_l/mc_R,\ l/\frac{1}{2}\rho V^2 mc_R\right)$

e distance between center of pressure and elastic axis of wing

e_R value of e at root $(e^* = e/e_R)$

G modulus of rigidity

GI torsional rigidity of wing section

I_R value of I at root $(I^* = I/I_R)$

S projected wing area

μ dimensionless constant $(mc_R/4b)$

β torsional-divergence parameter $\left(\sqrt{\frac{m\rho}{8G}\frac{e_R}{c_R}\frac{bc_R}{\sqrt{I_R}}}\,V\right)$

y spanwise coordinate measured from wing root in units of semispan

G,H,g auxiliary functions defined in equations (86), (87), and (91)

γ,a constants defining taper characteristics of wing (equation (16))

y_1 auxiliary variable (1 − ay)

ν,δ,λ constants defined in equations (20), (21), and (24)

A,B,C arbitrary constants

σ weighting coefficient of Simpson's rule

sf corresponding to section-force theory (as subscript)

MATHEMATICAL FORMULATION OF THE PROBLEM

If a wing with an initial angle-of-attack distribution $\alpha_0(y)$ is subjected to a lift distribution $l(y)$ per unit span and the resultant air force $l dy$, associated with a spanwise element of the wing acts at a distance from the elastic axis of the wing, a change of

angle of attack ϑ takes place due to the torsional flexibility of the wing. (See fig. 6.) The relationship between the additional angle of attack and the lift per unit span is given approximately by the differential equation

$$\frac{d}{d\left(\frac{b}{2}y\right)}\left[GI(y)\frac{d\vartheta}{d\left(\frac{b}{2}y\right)}\right] = -\,e(y)\;l(y) \qquad (1)$$

where y is a spanwise coordinate measured from the wing root in units of half span $b/2$, e is the distance between the center of pressure and the elastic axis, and GI is the torsional rigidity. It is known that this approximate equation neglects the effect of the spanwise variation of twist on the stresses and deformations of the wing, and that in a more accurate theory equation (1) would be replaced by a fourth-order differential equation for ϑ. Since, however, all calculations of the divergence velocity by means of the section-force theory of which the authors have knowledge are based on equation (1) and the main purpose of this paper is the estimation of the aerodynamic span effect, it is thought that satisfactory results may be obtained if the present calculations also are based on this equation. It may be stated that there are no essential difficulties in extending the work of this paper in the direction of refined procedures for the determination of the elastic deformations.

An additional relationship is afforded by the lifting-line integral equation

$$l(y) = mc(y)\left\{\frac{1}{2}\rho V^2\ (\alpha_0 + \vartheta) - \frac{1}{4\pi b}\,⨍_{-1}^{1}\frac{dl}{d\eta}\,\frac{d\eta}{y-\eta}\right\} \qquad (2)$$

where c is the chord of the wing, ρ is the density of air, V is the velocity of flight, and m is a profile constant. The notation $⨍$ is used in equation (2) to indicate that the Cauchy principal value of the integral is to be taken, according to the definition

$$⨍_{-1}^{1} = \lim_{\epsilon\to 0}\left\{\int_{-1}^{y-\epsilon} + \int_{y+\epsilon}^{1}\right\}$$

If an auxiliary lift function $F(y)$ is defined as

$$F(y) = \frac{l(y)}{\frac{1}{2}\rho V^2 m\ c_R} = \frac{c(y)}{m\ c_R} c_l(y) \qquad (3)$$

where $c_l(y)$ is the conventional section-lift coefficient, equations (1) and (2) can be written in the form

$$\frac{d}{dy}\left[I^*(y)\frac{d\phi}{dy}\right] + \beta^2\ e^*(y)\ F(y) = 0 \qquad (4)$$

$$\frac{F(y)}{c^*(y)} + \frac{\mu}{\pi}\int_{-1}^{1} \frac{dF}{d\eta}\ \frac{d\eta}{y-\eta} = \alpha_o(y) + \phi(y) \qquad (5)$$

where μ is a dimensionless parameter

$$\mu = \frac{m}{4}\ \frac{c_R}{b} \qquad (6)$$

and β^2, is defined by the relationship

$$GI_R\beta^2 = \frac{1}{2}\rho V^2 m \left(\frac{b}{2}\right)^2 e_R\ c_R \qquad (7)$$

In these equations c_R represents the root chord while $c^*(y)$ is the ratio of the chord to the root chord,

$$c^*(y) = \frac{c(y)}{c_R}$$

and analogous definitions apply to e_R, e^*, I_R and I^*.

If the wing is restrained from twisting at the root ($y = 0$), and no end twisting moments are applied, the boundary conditions for the function ϕ are

$$\left.\begin{aligned} \vartheta(0) &= 0 \\ \left[I\,\frac{d\vartheta}{dy}\right]_{y=\pm 1} &= 0 \end{aligned}\right\} \qquad (8)$$

while vanishing of the lift at the wing tips leads to the boundary conditions

$$F(\pm 1) = 0 \qquad (9)$$

The determination of continuous functions F and ϑ satisfying the simultaneous equations (4) and (5), together with the boundary conditions of equations (8) and (9), constitutes the basic problem of torsional divergence. It should be remarked that unless ϑ is an odd function of y the three boundary conditions of equation (8) do not, in general, permit a regular solution for ϑ. The physical explanation of this occurrence lies in the fact that unless the wing loading is antisymmetrical with respect to the wing root the restraint offered by the fuselage, in this formulation of the problem, is equivalent to a concentrated twisting moment, so that the function $I\,\frac{d\vartheta}{dy}$ must have a discontinuity at the root $(y = 0)$. Also, since it is necessary that the function $F(y)$ have a continuous derivative at interior points of the span in order that the left-hand side of equation (5) be continuous, it follows that the function

$$\frac{\mu}{\pi} \oint_{-1}^{1} \frac{dF}{d\eta} \frac{d\eta}{y-\eta}$$

which represents the so-called "induced angle of attack" must have a discontinuous first derivative at the wing root.

Equations (4) and (5) can be combined into a single integro-differential equation by the elimination of the function ϑ. Thus, if equation (4) is integrated twice and the boundary conditions of equation (8) are imposed, an expression for ϑ can be determined in the form

$$\phi(y) = \beta^2 \int_{-1}^{1} G(y,\eta)e^*(\eta)F(\eta)d\eta \qquad (10)$$

where the function G depends upon the function I^*, and the introduction of equation (10) into equation (5) gives the form

$$\frac{F(y)}{c^*(y)} + \frac{\mu}{\pi}\int_{-1}^{1}\frac{dF}{d\eta}\frac{d\eta}{y-\eta} - \beta^2\int_{-1}^{1}G(y,\eta)e^*(\eta)F(\eta)d\eta = \alpha_0(y) \qquad (11)$$

If the function α_0 is not identically zero, equation (11), together with the boundary conditions of equation (9), possesses, in general, a unique solution $F(y)$. It is known, however, that there exists an infinite set of critical values of the parameter β for which no solution to equation (11) exists. In addition, as β approaches one of these critical values, the magnitude of the corresponding functions $F(y)$ and $\phi(y)$ increases without limit. Since β is proportional to the velocity of flight, the critical values of β correspond to critical velocities at which a very large (theoretically infinite) twisting force is experienced by the wing. Thus an accurate determination of the smallest critical value of β is desirable for purposes of structural wing design. The value of V corresponding to the smallest critical valuesof β is designated as the torsional-divergence velocity.

According to the theory of integro-differential equations the critical values of β for which no solution to equation (11) exists are identical with the values of β for which the homogeneous equation, with α_0 identically zero, possesses a solution. That is, a critical value of β corresponds to such a critical velocity of flight that an initially untwisted airfoil $(\alpha_0 \equiv 0)$ may become deformed. In the present linearized theory the magnitude of the deflection in this case is of undetermined magnitude because of the homogeneity of equation (11) when $\alpha_0 = 0$.

The previous explicit determination of the critical values of the parameter β were based on the so called

"section-force theory," which disregards the aerodynamic effect of finite span by neglecting the integral representing the induced angle of attack in equation (5) (reference 1). In what follows, a brief treatment of the section-force procedure is first given, after which a method of successive approximation is presented for the determination of the torsional-divergence flight velocity and of the form of the corresponding deflection and lift curves, according to the lifting-line theory of equations (4) and (5). Since the magnitude of the torsional-divergence velocity is independent of the initial angle-of-attack distribution, it will be assumed that the wing is initially at a zero angle of attack ($\alpha_0(y) = 0$).

SOLUTION OF THE PROBLEM ACCORDING TO SECTION-FORCE THEORY

If the integral in equation (5) is neglected and an initial zero angle of attack is assumed, equations (4) and (5) become

$$\frac{d}{dy}\left[I^*(y)\,\frac{d\vartheta}{dy}\right] + \beta^2 e^*(y)F(y) = 0 \qquad (12)$$

$$F(y) = c^*(y)\vartheta(y) \qquad (13)$$

These equations, together with the boundary conditions

$$\left.\begin{aligned} \vartheta(0) &= 0 \\ \left[I\,\frac{d\vartheta}{dy}\right]_{y=\pm 1} &= 0 \end{aligned}\right\} \qquad (14)$$

are taken as the basis of the analysis of the problem of torsional divergence according to the section-force theory. The boundary conditions of equation (9) are not prescribed in this theory.

Introducing equation (13) into equation (12) leads to a homogeneous differential equation in ϑ

$$\frac{d}{dy}\left[I^{*}(y)\,\frac{d\theta}{dy}\right] + \beta^{2} c^{*}(y)\, e^{*}(y)\, \theta(y) = 0 \tag{15}$$

which together with the homogeneous boundary conditions of equation (14) is sufficient to determine the critical values of the parameter β and the corresponding critical deflection modes. (See reference 1.)

A Class of Explicit Solutions

The integration of equation (15) in closed form is possible, in particular, in cases when the chord and section stiffness vary according to the laws

$$\left.\begin{aligned} c^{*}(y) = e^{*}(y) &= (1 - ay)^{\gamma_1} \\ I^{*}(y) &= (1 - ay)^{\gamma_2} \end{aligned}\right\} \tag{16}$$

where γ_1 and γ_2 are arbitrary positive constants and a is a positive constant less than unity. With the substitution

$$1 - ay = y_1 \tag{17}$$

equation (15) becomes

$$\frac{d}{dy_1}\left[y_1^{\gamma_2}\,\frac{d\theta}{dy_1}\right] + \left(\frac{\beta}{a}\right)^{2} y_1^{2\gamma_1}\, \theta = 0 \tag{18}$$

and if

$$\gamma_2 \neq 2(\gamma_1 + 1)$$

the solution is known to be of the form

$$\theta(y) = y_1^{\frac{1-\gamma_2}{2}}\, Z\left(\delta\,\frac{\beta}{a}\, y_1^{1/\delta}\right) \tag{19}$$

where Z_ν is the general Bessel function of order ν and

$$\nu = \frac{1 - \gamma_2}{2\gamma_1 - \gamma_2 + 2} \tag{20}$$

and

$$\delta = \frac{2\nu}{1 - \gamma_2} \tag{21}$$

In the special cases when

$$\gamma_2 = 2\,(\gamma_1 + 1) \tag{22}$$

the solution can be expressed in the form

$$\vartheta(y) = y_1^{-\frac{2\gamma_1+1}{2}} \left\{ A \sin(\lambda \log y_1) + B \cos(\lambda \log y_1) \right\} \tag{23}$$

where A and B are arbitrary constants and

$$\lambda = \sqrt{\left(\frac{\beta}{a}\right)^2 - \left(\frac{2\gamma_1 + 1}{2}\right)^2} \tag{24}$$

The solution is evaluated explicitly in the following four cases:

1. Uniform chord, uniform stiffness ($\gamma_1 = \gamma_2 = 0$).- In this case equation (15) becomes

$$\frac{d^2\vartheta}{dy^2} + \beta^2 \vartheta = 0 \tag{25}$$

and the general continuous solution having continuous derivatives except at $y = 0$ is of the form

$$\vartheta(y) = A \sin \beta|y| + B \sin \beta y + C \cos \beta y \tag{26}$$

where the first term has a discontinuous derivative at the root. The boundary conditions of equation (14) require that

$$C = 0 \tag{27}$$

and

$$\cos \beta = 0 \tag{28}$$

Equation (28) indicates that equations (25) and (14) possess solutions only when

$$\beta = \frac{(2n + 1)\pi}{2} \tag{29}$$

where n is an integer. The smallest of these values, $\beta = \pi/2$, then corresponds to the torsional-divergence velocity, and the corresponding deflection mode is of the form

$$\vartheta(y) = A \sin \beta |y| + B \sin \beta y \tag{30}$$

where A and B are arbitrary constants.

Equation (30) shows that this mode may have both symmetrical and antisymmetrical components, so that according to this theory the two halves of the wing deflect independently of each other. Moreover, according to the section-force theory the symmetrical and antisymmetrical deflection modes correspond to the same critical flight velocity. For this reason it will be convenient in this section to consider only one-half of the wing. The solution of the problem for the first deflection mode then can be written in the form

$$\left.\begin{aligned} \beta &= \frac{\pi}{2} \\ F(y) = \vartheta(y) &= A \sin \frac{\pi}{2} y \end{aligned}\right\} \tag{31}$$

3. Uniform chord, quadratically decreasing stiffness ($\gamma_1 = 0$, $\gamma_2 = 2$).- If only one-half of the wing is considered, the general solution of equation (15) is obtained from equation (23) in the form (see also reference 1)

$$\vartheta(y) = \frac{1}{\sqrt{y_1}}\left\{A \sin (\lambda \log y_1) + B \cos (\lambda \log y_1)\right\} \qquad (32)$$

where

$$y_1 = 1 - ay \qquad (17)$$

and

$$\lambda = \sqrt{\left(\frac{\beta}{a}\right)^2 - \frac{1}{4}} \qquad (33)$$

The boundary conditions of equation (14) then require that

$$B = 0 \qquad (34)$$

and

$$\tan\left[\lambda \log (1 - a)\right] = 2\lambda \qquad (35)$$

Equation (35) has an infinite number of solutions which in conjunction with equation (33) determine the critical values of β for which a solution to the problem exists. A numerical evaluation of the solution is presented for two degrees of taper:

$$\text{(a)} \quad \frac{\text{tip stiffness}}{\text{root stiffness}} = \frac{1}{4} \quad \left(a = \frac{1}{2}\right)$$

In this case the smallest root of equation (35), which becomes

$$\tan (\lambda \log 2) + 2\lambda = 0$$

is found by a conventional method of successive approximations to be

$$\lambda = 2.546$$

and equation (33) then gives the corresponding critical value of β,

$$\beta = 1.297$$

The first mode of deflection for one-half the wing is thus described by the equations

$$\left.\begin{aligned} \beta &= 1.297 \\ F(y) = \vartheta(y) &= \frac{A}{\sqrt{1 - \frac{1}{2}y}} \sin\left[2.546 \log\left(1 - \frac{1}{2}y\right)\right] \end{aligned}\right\} \qquad (36)$$

(b) $\dfrac{\text{tip stiffness}}{\text{root stiffness}} = \dfrac{1}{36} \quad \left(a = \dfrac{5}{6}\right)$

For this case, which was evaluated in reference 1, the solution for the first mode is obtained in the form

$$\left.\begin{aligned} \beta &= 1.016 \\ F(y) = \vartheta(y) &= \frac{A}{\sqrt{1 - \frac{5}{6}y}} \sin\left[1.112 \log\left(1 - \frac{5}{6}y\right)\right] \end{aligned}\right\} \qquad (37)$$

3. Linear chord, quadratically decreasing stiffness ($\gamma_1 = 1$, $\gamma_2 = 2$).- The solution of equation (15) is given by equation (19) where $\nu = -1/2$ and $s = 1$. The Bessel functions of order $\pm 1/2$ are expressible in terms of the circular functions, so that the general solution of equation (15) can be written in the form

$$\vartheta(y) = \frac{1}{y_1}\left(A \sin \frac{\beta}{a} y_1 + B \cos \frac{\beta}{a} y_1\right) \qquad (38)$$

If the boundary condition $\vartheta(0) = 0$ is imposed, it follows that

$$\vartheta(y) = \frac{A}{y_1} \sin \frac{\beta}{a} (1 - y_1) = \frac{A}{1 - ay} \sin \beta y \qquad (39)$$

The boundary condition $\vartheta'(1) = 0$ determines the critical values of β as the roots of the equation

$$\tan \beta + \frac{1 - a}{a} \beta = 0 \tag{40}$$

Equations (13) and (39) determine the auxiliary lift function F in the form

$$F(y) = A \sin \beta y \tag{41}$$

In particular, if $a = 1/2$, so that the tip chord is one-half the root chord and the tip stiffness is one-fourth the root stiffness, the first deflection mode of half the wing is described by the equations

$$\left.\begin{aligned} \beta &= 2.029 \\ F(y) &= A \sin 2.029\, y \\ \theta(y) &= \frac{A}{1 - \frac{1}{2}y} \sin 2.029\, y \end{aligned}\right\} \tag{42}$$

4. Linear chord, quartically decreasing stiffness ($\gamma_1 = 1$, $\gamma_2 = 4$). - The solution of equation (15) is given by equation (23) in the form

$$\theta(y) = \frac{1}{{y_1}^{3/2}} \left\{ A \sin (\lambda \log y_1) + B \cos (\lambda \log y_1) \right\} \tag{43}$$

where

$$\lambda = \sqrt{\left(\frac{\beta}{a}\right)^2 - \frac{9}{4}} \tag{44}$$

while the boundary conditions of equation (14) are satisfied if

$$B = 0 \tag{45}$$

and the parameter λ is a solution of the equation

$$\tan \left[\lambda \log (1 - a)\right] = \frac{2}{3} \lambda \tag{46}$$

In the case $a = 1/2$, where the tip chord is one-half the root chord and the tip stiffness is one-sixteenth the root stiffness, the following data are obtained for the first deflection mode of half the wing:

$$\left.\begin{aligned} \beta &= 1.653 \\ F(y) &= \frac{A}{\sqrt{1 - \frac{1}{2} y}} \sin\left[2.946 \log\left(1 - \frac{1}{2} y\right)\right] \\ \vartheta(y) &= \frac{A}{\left(1 - \frac{1}{2} y\right)^{3/2}} \sin\left[2.946 \log\left(1 - \frac{1}{2} y\right)\right] \end{aligned}\right\} \tag{47}$$

Solution by a Method of Successive Approximations

The determination of the smallest critical value of the parameter β, and of the corresponding functions F and ϑ from equations (12) and (13), in cases when the solution of equation (15) in closed form is not readily obtained, is conveniently accomplished by a method of successive approximations similar to a method associated with the names of Stodola and Vianello.

A convenient function $\vartheta_1(y)$ is first chosen in such a way that it satisfies the boundary conditions of equation (14). A function $F_1(y)$ is then determined by introducing this function into equation (13).

$$F_1(y) = c^*(y)\,\vartheta_1(y) \tag{48}$$

If the function F_1 is introduced into equation (12), the resultant equation

$$\frac{d}{dy}\left[I^*(y)\,\frac{d\vartheta}{dy}\right] = -\beta^2 e^*(y)\,F_1(y) \tag{49}$$

can be solved for ϑ by direct integration. Suppose that the solution of this equation satisfying the boundary conditions of equation (14) is given by

$$\bar{\vartheta}_1(y) = \beta^2 \varphi_1(y) \quad (50)$$

If $\vartheta_1(y)$ were the exact solution of the problem corresponding to a critical value of β, it would be possible to choose β so that the functions ϑ_1 and $\bar{\vartheta}_1$ are identical. When this is not the case, an approximation to the desired critical value of β can be determined if it is required that the functions ϑ_1 and $\bar{\vartheta}_1$ agree as well as possible over the interval $|y| \leqslant 1$. This determination is usually accomplished by requiring that the two functions coincide at a suitably chosen point $y = y_0$, so that a first approximation to β is given by

$$\beta_1^2 = \frac{\vartheta_1(y_0)}{\varphi_1(y_0)} \quad (51)$$

A more accurate procedure proposed here consists in requiring that the integral, over the span, of the difference between the functions ϑ_1 and $\bar{\vartheta}_1$ vanish, so that a first approximation to β is given by

$$\beta_{I1}^2 = \frac{\int_{-1}^{1} \vartheta_1(y)\,dy}{\int_{-1}^{1} \varphi_1(y)\,dy} \quad (52)$$

If a convenient multiple of $\bar{\vartheta}_1(y)$ is treated as a second approximation, $\vartheta_2(y)$, the process can now be repeated indefinitely, and it can be shown that the limiting value

$$\beta^2 = \lim_{n \to \infty} \frac{\int_{-1}^{1} \vartheta_n(y)\,dy}{\int_{-1}^{1} \varphi_n(y)\,dy}$$

gives the smallest critical value of β^2 while the limiting function

$$\vartheta(y) = \lim_{n\to\infty} \vartheta_n(y)$$

represents the corresponding twisting mode of the wing. Since the amplitude of the function ϑ is indeterminate within the framework of the linear theory, it is convenient at the beginning of each cycle to magnify the approximation $\bar{\vartheta}_{n-1}$ given by the preceding cycle so that the initial approximation ϑ_n in each cycle has a maximum amplitude of unity.

Rapid convergence of the process has been found if the initial approximation $\vartheta_1(y)$ is defined as a suitable multiple of the solution of equation (12) corresponding to a uniform distribution of lift along the span, so that

$$\frac{d}{dy}\left[I^*(y)\,\frac{d\vartheta_1}{dy}\right] = (\text{const.}) \times e^*(y) \tag{53}$$

As an illustration of this procedure, the special case of a uniform wing for which

$$c^*(y) = e^*(y) = I^*(y) = 1 \tag{54}$$

is analysed. As before, it is sufficient to consider only one-half the wing. The initial approximation to ϑ is determined by replacing $F(y)$ by a constant in equation (12), so that

$$\frac{d^2\vartheta_1}{dy^2} = \text{constant} \tag{55}$$

The solution of this equation satisfying the conditions

$$\vartheta(0) = \vartheta'(1) = 0 \tag{56}$$

is found to be

$$\vartheta_1(y) = (\text{Const.}) \times \left(y - \frac{1}{2}\,y^2\right) \tag{57}$$

The constant is arbitrary and is conveniently chosen so that $\vartheta_1(1) = 1$. It then follows that

$$\vartheta_1(y) = 2y - y^2 \tag{58}$$

From equation (48) there follows

$$F_1(y) = \vartheta_1(y) = 2y - y^2 \tag{59}$$

and the introduction of $F_1(y)$ into equation (49) yields the equation

$$\frac{d^2\overline{\vartheta}_1}{dy^2} = \beta^2(y^2 - 2y) \tag{60}$$

Integrating equation (60) twice and imposing the boundary conditions of equation (56), the function $\overline{\vartheta}_1$ is determined as

$$\overline{\vartheta}_1(y) = \beta^2\left(\frac{2}{3}y - \frac{1}{3}y^3 + \frac{1}{12}y^4\right) \tag{61}$$

The condition

$$\int_0^1 \vartheta_1(y)\,dy = \int_0^1 \overline{\vartheta}_1(y)\,dy \tag{62}$$

then gives the first approximation to the critical value of β,

$$\left.\begin{aligned} \beta_1^2 &= 2.5 \\ \beta_1 &= 1.5811 \end{aligned}\right\} \tag{63}$$

which differs from the exact value, $\beta = \frac{\pi}{2} = 1.5708$, by 0.65 percent.

If the cycle is repeated, starting with the initial approximation

$$\vartheta_2(y) = \frac{\overline{\vartheta}_1(y)}{\overline{\vartheta}_1(1)} = \frac{1}{5}(8y - 4y^3 + y^4) \tag{64}$$

it is found that

$$\bar{\vartheta}_2(y) = \beta^2 \left(\frac{16}{25} y - \frac{4}{15} y^3 + \frac{1}{25} y^5 - \frac{1}{150} y^6\right) \qquad (65)$$

and the condition

$$\int_0^1 \vartheta_2(y)\, dy = \int_0^1 \bar{\vartheta}_2(y)\, dy \qquad (66)$$

gives a second approximation to β,

$$\left.\begin{aligned} \beta_2^2 &= \frac{42}{17} \\ \beta_2 &= 1.5718 \end{aligned}\right\} \qquad (67)$$

which differs from the exact value by less than 0.07 percent.

Since the initial assumption for ϑ in each cycle was so defined that its maximum value (at $y = 1$) is unity, an estimate of the rate of convergence is afforded by comparing with unity the maximum values of the approximations $\bar{\vartheta}_n$ obtained after successive cycles. Thus, in the present example, equations (61) and (63) give

$$\bar{\vartheta}_1(1) = \frac{5}{2} \times \frac{5}{12} = 1.042$$

and equations (65) and (67) give

$$\bar{\vartheta}_2(1) = \frac{42}{17} \times \frac{61}{150} = 1.005$$

If the successive approximations to β were determined by the conditions

$$\bar{\vartheta}_n(1) = \vartheta_n(1) = 1 \qquad (68)$$

in place of the integral conditions used here, the values

$$\left.\begin{aligned} \beta_1 &= 1.5492 \\ \beta_2 &= 1.5681 \end{aligned}\right\} \tag{69}$$

which differ from the exact value by 1.38 percent and 0.17 percent, respectively, would be obtained.

A further illustration of the advantage of the integral condition (equation (52)) over the condition usually imposed (equation (51)) is afforded by a consideration of the fourth case treated in the earlier part of this section (equation (47)). In this case the initial approximation to ϑ is of the form

$$\vartheta_1(y) = -\frac{1}{2}\left(y_1^{-3} - 3y_1^{-2} + 2\right) \tag{70}$$

where

$$y_1 = 1 - \frac{1}{2}y \tag{71}$$

and, using equation (48), equation (49) becomes

$$\frac{d}{dy_1}\left[y_1^4 \frac{d\bar{\vartheta}_1}{dy_1}\right] = \frac{1}{2}\beta^2\left(y_1^{-1} - 3 + 2y_1^2\right) \tag{72}$$

If equation (72) is solved, subject to the boundary conditions of equation (56), and the result, together with equation (70), is introduced into equation (62) the first approximation to β is obtained as

$$\beta_1 = 1.664 \tag{73}$$

This value differs from the exact value $\beta = 1.653$ given in equation (47) by about 0.66 percent. If, in place of equation (62), the equation

$$\bar{\vartheta}_1(1) = 1 \tag{74}$$

is used for the determination of β, an approximation

$$\beta_1 = 1.612 \tag{75}$$

is obtained which differs from the exact value by about 2.48 percent.

PROCEDURE FOR SOLUTION OF THE PROBLEM ACCORDING TO LIFTING-LINE THEORY

While a direct treatment of equation (11) by approximate methods is possible, it is more convenient, particularly in dealing with the critical deflection modes, to work with equations (4) and (5) and to proceed by a method of successive approximations similar to the procedure given in the preceding section. According to the lifting-line theory the symmetrical and antisymmetrical deflection modes correspond, in general, to different critical flight velocities. In what follows, the treatment is restricted to the symmetrical case and treatment of the antisymmetrical case is left for future work. Attention then may be restricted as before to one-half the span $(0 < y < 1)$. Also, as in the preceding section, it is assumed that the wings considered are initially at a zero angle of attack.

Starting with an initial assumption $\vartheta_1(y)$, determined as before as the deflection corresponding to a uniform distribution of lift along the span, according to the differential equation

$$\frac{d}{dy}\left[I^*(y)\,\frac{d\vartheta_1}{dy}\right] = (\text{const.}) \times e^*(y) \qquad (76)$$

and the boundary conditions $\vartheta_1(0) = I^*(1)\,\vartheta_1{}'(1) = 0$, a function $F_1(y)$ is next defined by equation (5),

$$\frac{F_1(y)}{c^*(y)} + \frac{\mu}{\pi}\int_{-1}^{1}\frac{dF_1}{d\eta}\,\frac{d\eta}{y-\eta} = \vartheta_1(y) \qquad (77)$$

and the boundary conditions

$$F_1(\pm 1) = 0 \qquad (78)$$

An approximate solution of this equation is conveniently obtained by a procedure given in reference 5, wherein an approximation to $F_1(y)$ is assumed in the form

$$F_1(y) \approx By^2 \log\left(\frac{1+\sqrt{1-y^2}}{|y|}\right) + \sum_{n=0}^{3} A_{2n}\, y^{2n}\sqrt{1-y^2} \qquad (79)$$

and the parameters B, $A_0, \ldots, A_6$ are determined by a method of least squares. The first term of the approximation of equation (79) is included since, as is shown in reference 5, the contribution of that term to the induced angle of attack has the required discontinuity in its first derivative at the root $(y = 0)$.

If the function $F_1(y)$ determined from equation (77) is introduced into equation (4), the resultant equation

$$\frac{d}{dy}\left[I^*(y)\,\frac{d\bar{\vartheta}_1}{dy}\right] = -\beta^2 e^*(y)\,F_1(y) \qquad (80)$$

subject to the boundary conditions of equation (8) can be solved for $\bar{\vartheta}_1$ by direct integration, after which the first approximation β_1 to the critical value of β is determined as before by the condition

$$\int_0^1 \vartheta_1(y)\,dy = \int_0^1 \bar{\vartheta}_1(y)\,dy \qquad (81)$$

A check on the accuracy of this approximation can be had by comparing the maximum values of the functions ϑ_1 and $\bar{\vartheta}_1$. Thus if ϑ_1 is chosen in such a way that

$$\vartheta_1(1) = 1 \qquad (82)$$

the degree of approximation attained is indicated by the closeness of the approximation

$$\bar{\vartheta}_1(1) \approx 1 \qquad (83)$$

If necessary, the process can be repeated indefinitely. However, in all the problems considered in this paper satisfactory results are afforded by a single cycle of operations. That is, the initial approximation ϑ_1

determined from equation (76) is sufficiently similar to the exact deflection mode that if the condition of equation (81) is satisfied, the two functions ϑ_1 and $\bar{\vartheta}_1$ agree closely over the entire span.

In case only a first approximation is required, the value of the integral $\int_0^1 \bar{\vartheta}_1(y)\,dy$ which is needed in the determination of β_1 can be found without explicitly determining the function $\vartheta_1(y)$. If equation (4) is integrated twice and the boundary conditions

$$\vartheta(0) = I^*(1)\,\vartheta'(1) = 0 \tag{84}$$

are satisfied, the resultant expression for ϑ can be written in the form

$$\vartheta(y) = \beta^2 \int_0^1 G(y,\eta)\; e^*(\eta)\; F(\eta)\; d\eta \tag{85}$$

where the Green's function G is defined by the equations

$$G(y,\eta) = \begin{cases} g(\eta) & 0 \leq \eta \leq y \\ \\ g(y) & y \leq \eta \leq 1 \end{cases} \tag{86}$$

and

$$g(y) = \int_0^y \frac{d\eta}{I^*(\eta)} \tag{87}$$

It then follows from equation (85) that

$$\int_0^1 \bar{\vartheta}_1(y)\,dy = \beta^2 \int_0^1 \left\{ \int_0^1 G(y,\eta)\,dy \right\} e^*(\eta)\; F_1(\eta)\; d\eta \tag{88}$$

and since

$$\int_0^1 G(y,\eta)\,dy = \int_0^\eta g(y)\,dy + g(\eta)\int_\eta^1 dy$$

$$= \int_0^\eta (1-\xi)\,g'(\xi)\,d\xi$$

$$= \int_0^\eta \frac{1-\xi}{I^*(\xi)}\,d\xi \tag{89}$$

equation (88) can be written in the form

$$\int_0^1 \bar{\vartheta}_1(y)\,dy = \beta_1^2\int_0^1 H(y)\,e^*(y)\,F_1(y)\,dy \tag{90}$$

where

$$H(y) = \int_0^y \frac{1-\eta}{I^*(\eta)}\,d\eta \tag{91}$$

If equation (90) is introduced into equation (81), the relationship determining the first approximation to β can be written in the form

$$\beta_1^2 = \frac{\int_0^1 \vartheta_1(y)\,dy}{\int_0^1 H(y)\,e^*(y)\,F_1(y)\,dy} \tag{92}$$

Equation (83), which affords a check on the accuracy obtained, can also be expressed in terms of the function F_1 if it is noticed that, from equation (85),

$$\bar{\vartheta}_1(1) = \beta_1^2\int_0^1 G(1,\eta)\,e^*(\eta)\,F_1(\eta)\,d\eta$$

$$= \beta_1^2\int_0^1 g(\eta)\,e^*(\eta)\,F_1(\eta)\,d\eta \tag{93}$$

Thus equation (83) can be written in the form

$$\bar{\vartheta}_1(1) = \beta_1 \int_0^1 g(y)\, c^*(y)\, F_1(y)\, dy \approx 1 \tag{94}$$

where the function g is defined by equation (87).

The procedure for obtaining a first approximation to the smallest critical value of β corresponding to a symmetrical deflection mode can be summarized as follows:

(1) Determine $\vartheta_1(y)$ from equation (76) and the boundary conditions $\vartheta_1(0) = I^*(1)\,\vartheta_1''(1) = 0$, and determine the multiplicative constant so that $\vartheta_1(1) = 0$.

(2) Determine $F_1(y)$ from equation (77) and the boundary conditions $F_1(\pm 1) = 0$ by the procedure of reference 5.

(3) Determine β_1 from equation (92). A check on the accuracy of the determination is provided by equation (94).

If greater accuracy is desired, the function $\bar{\vartheta}_1(y)$ is next determined from equation (80) and the boundary conditions $\bar{\vartheta}_1(0) = I^*(1)\,\bar{\vartheta}_1'(1) = 0$. A function $\vartheta_2(y)$ is then defined as

$$\vartheta_2(y) = \frac{\bar{\vartheta}_1(y)}{\bar{\vartheta}_1(1)}$$

so that $\vartheta_2(1) = 1$, the corresponding function $F_2(y)$ is determined as in step (2) of the preceding paragraph, and a second approximation to β is determined as in step (3).

It may be remarked that the entire process can be carried out conveniently by numerical or mechanical methods. If the solution of equation (77) is found by the procedure of reference 5, values of the initial approximation ϑ_1 and of the chord function c^* are needed only at nine equally spaced points along the semispan. The

values of the function F_1 at the same nine points are then determined from the data presented in reference 5 by a purely numerical procedure requiring less than two hours of time. If the values of the functions e^*, H, and g are tabulated at these points, the integrals needed in equations (92) and (94) can be readily evaluated by Simpson's rule, modified if necessary so as to take into account the fact that the function $F_1(y)$ has an infinite derivative at the wing tip $(y = 1)$. If the approximation of equation (79) is used, such a modified formula, derived in the appendix, is of the form

$$\int_0^1 \varphi(y)\, F_1(y)\, dy \approx \sum_{k=0}^{8} \sigma_k\, \varphi(x_k)\, F_1(x_k) + 0.00506 \left\{ B + \sum_{n=0}^{3} A_{2n} \right\} \varphi(1) \tag{95}$$

where $y_k = k/8$ and σ_k is the Simpson's-rule weighting coefficient associated with the point y_k.

$$\left. \begin{aligned} &\sigma_0 = \frac{1}{24} \quad \sigma_1 = \frac{4}{24} \quad \sigma_2 = \frac{2}{24} \quad \sigma_3 = \frac{4}{24} \quad \sigma_4 = \frac{2}{24} \\ &\sigma_5 = \frac{4}{24} \quad \sigma_6 = \frac{2}{24} \quad \sigma_7 = \frac{4}{24} \quad \sigma_8 = \frac{1}{24} \end{aligned} \right\} \tag{96}$$

The last term of equation (95) gives the correction due to the fact that $F_1(y)$ has an infinite derivative at the point $y = 1$.

LIFTING-LINE ANALYSIS OF EXPLICIT CASES

In order to illustrate the procedure of the preceding section and to investigate the importance of the aerodynamic span effect, the cases analyzed in a preceding section according to the section-force theory are now reconsidered on the basis of the lifting-line theory. Since only symmetrical deflections are to be treated, attention will be restricted to one-half the wing.

In the numerical calculations it is assumed that

$$\mu = \frac{m}{4}\frac{c_R}{b} = \frac{1}{4} \tag{97}$$

For an untapered wing this value of μ corresponds to an aspect ratio

$$\frac{b}{c} = m \ (\approx 6) \tag{98}$$

while for the general case the corresponding aspect ratio b^2/S, where S is the projected wing area, is given by

$$\frac{b^2}{S} = \frac{m}{\frac{1}{2}\int_{-1}^{1} c^*(y)\,dy} \tag{99}$$

In particular, for a symmetrical wing with linearly tapering chord,

$$c^*(y) = \frac{c(y)}{c_R} = 1 - a\,|y| \tag{100}$$

it follows that

$$\frac{b^2}{S} = \frac{m}{1 - \frac{1}{2}a} \tag{101}$$

It should be remarked that the calculations contained in this section were made by retaining a larger number of significant figures than are included in the data presented. Thus if the calculations are repeated on the basis of the tabulated data, the results may differ slightly from the results given here.

1. Uniform chord, uniform stiffness ($c^* = e^* = I^* = 1$).

The auxiliary functions H and g are determined from equations (91) and (87) in the form

$$H(y) = y - \frac{1}{2} y^2 \tag{102}$$

and

$$g(y) = y \tag{103}$$

As in a preceding section, the initial approximation ϑ_1 to the first symmetrical mode (for $0 \leq y \leq 1$) is determined from equation (76) as

$$\vartheta_1(y) = 2y - y^2 \tag{104}$$

Next the function F_1 is to be determined from equation (77) which becomes

$$F_1(y) + \frac{1}{4\pi} \oint_{-1}^{1} \frac{dF_1}{d\eta} \frac{d\eta}{y-\eta} = 2y - y^2 \tag{105}$$

If the approximation of equation (79) is assumed, the procedure of reference 5 determines the constants as follows:

$$\left.\begin{array}{lll} B = 0.4298 & A_0 = 0.2680 & A_2 = 1.2950 \\ A_4 = -2.3442 & A_6 = 1.8286 & \end{array}\right\} \tag{106}$$

The data needed for the computation of β_1 according to equation (92) and for the check calculation of equation (94) are listed in table 1(a). Thus, according to equation (92), the quantity $\beta_1{}^2$ is the ratio of two integrals of which the first is obtained by Simpson's rule as the weighted sum of entries in the fourth column:

$$\int_0^1 \vartheta_1(y)\, dy \approx \sum_{k=0}^{8} \sigma_k\, \vartheta_1(y_k) = 0.6667 \tag{107}$$

and the second is obtained in virtue of equation (95) as the sum of weighted products of corresponding entries in the second and fifth columns, plus a correction term:

$$\int_0^1 H(y)\ F_1(y)\ dy \approx \sum_{k=0}^{8} \sigma_k\ H(y_k)\ F_1(y_k)$$

$$+ (0.00506)(1.4773)(0.5000) = 0.1654 \qquad (108)$$

Equation (92) then gives

$$\left.\begin{aligned} \beta_1^2 &= 4.032 \\ \beta_1 &= 2.008 \end{aligned}\right\} \qquad (109)$$

In the same way, the integral contained in equation (94) is evaluated as the sum of weighted products of corresponding entries in the third and fifth columns of table I(a), plus a correction term:

$$\int_0^1 g(y)\ F_1(y)\ dy \approx \sum_{k=0}^{8} \sigma_k\ g(y_k)\ F_1\ (y_k)$$

$$+ (0.00506)(1.4773)(1.0000) = 0.2479 \qquad (110)$$

Equation (94) then gives

$$\overline{\vartheta}_1(1) = 0.9994 \qquad (111)$$

(While in the present case the integrals in equations (107), (108), and (110) can easily be evaluated directly, the numerical method of evaluation just outlined is particularly convenient in case the integrands have complicated analytical expressions or are determined graphically.)

Since equation (92) requires that the integral of the difference between the successive approximations ϑ_1 and $\overline{\vartheta}_1$ be zero and equation (111) shows that the two curves agree almost exactly at the tip and, further, since both functions vanish at the root and have zero

slope at the tip, it would appear that the agreement between the two functions over the span is such that the process need not be continued. The functions ϑ_1 and F_1 would then be considered as the deflection and lift modes corresponding to the critical value β_1 given in equation (109).

In order to verify the accuracy of this approximation the second approximation is now determined. The function $\bar{\vartheta}_1(y)$ is obtained from equation (80) by direct integration and is tabulated in the sixth column of table 1(a). If the values of this function are divided by the value of the function at the tip, the corresponding values of the function $\vartheta_2(y) = \bar{\vartheta}_1(y)/\bar{\vartheta}_1(1)$ are obtained and listed in table 1(b). The parameters specifying the function $F_2(y)$ which satisfies the equation

$$F_2(y) + \frac{1}{4\pi}\oint_{-1}^{1}\frac{dF_2}{d\eta}\frac{d\eta}{y-\eta} = \vartheta_2(y) \qquad (112)$$

are then found by the procedure of reference 5 as follows:

$$B = 0.3884 \qquad A_0 = 0.2624 \qquad A_2 = 1.4074$$
$$A_4 = -2.4319 \qquad A_6 = 1.8538 \qquad (113)$$

The values of the function F_2 are presented in table 1(b). From equation (92) the second approximation to β is found

$$\left.\begin{aligned}\beta_2^{\,3} &= 4.023\\ \beta_2 &= 2.006\end{aligned}\right\} \qquad (114)$$

and equation (94) gives the result

$$\bar{\vartheta}_2(1) = 1.0006 \qquad (115)$$

For final comparison, the values of the function $\bar{\vartheta}_2(y)$, so determined from $F_2(y)$ by an equation analogous to equation (80), are included in table 1(b).

It may be said that, contrary to expectations, a less rapid rate of convergence of the iterative process occurs if, instead of taking as the initial approximation to the deflection mode the solution of equation (76), the solution of the problem according to the section-force theory is taken as the initial approximation. The results of a calculation based on this procedure, with

$$\vartheta_1(y) = \vartheta_{sf}(y) = \sin\frac{\pi}{2}y \qquad (116)$$

are presented in table 1(c). While the value obtained for β_1,

$$\beta_1 = 2.004 \qquad (117)$$

and the values of the function $\vartheta_2(y)$ agree closely with the preceding results, appreciable differences are present between the successive approximations ϑ_1 and $\bar{\vartheta}_1$ in this procedure. An indication of the fact that the function given in equation (116) does not afford an accurate approximation to the actual critical deflection mode would be afforded, without a complete explicit evaluation of the function $\bar{\vartheta}_1$, by the readily calculated value of $\bar{\vartheta}_1(1)$ for this solution,

$$\bar{\vartheta}_1(1) = 0.9582 \qquad (118)$$

From these results three useful conclusions, which will be further substantiated in the following treatment, may be drawn:

(1) The approximate deflection mode determined as the solution of equation (76) is more nearly in agreement with the actual mode than is the mode predicted by the section-force theory.

(2) The present method of analysis is not highly sensitive to the choice of the initial approximation to the deflection mode as regards the determination of the first approximation to the critical value of β.

(3) If β_1 is determined by equation (92) the degree of approximation attained in equation (94) affords a reasonably accurate estimate of the agreement between the initial approximation ϑ_1 and the actual deflection mode.

If the results presented in equations (31) and (114) are compared, it is seen that the aerodynamic span effect is responsible for an increase of about 28 percent in the predicted value of the torsional-divergence velocity. In figure 1 there is presented a comparison of the successive approximations ϑ_1, ϑ_2, and ϑ_3 to the deflection mode, and of the successive approximations F_1 and F_2 to the corresponding lift-distribution function. The mode

$$\vartheta_{sf} = F_{sf} = \sin \frac{\pi}{2} y$$

predicted by the section-force theory is also included.

2. Uniform chord, quadratically decreasing stiffness $\left[c^* = e^* = 1,\ I^* = (1 - ay)^2\right]$.- The functions ϑ_1, H, and g are obtained from equations (76), (91), and (87) in the form

$$\vartheta_1(y) = \frac{1}{\log\left(\frac{1}{1 - a}\right) - a}\left\{\log \frac{1}{y_1} - (1 - a)\left(\frac{1}{y_1} - 1\right)\right\} \tag{119}$$

$$H(y) = \frac{1}{a^2}\left\{\log \frac{1}{y_1} - (1 - a)\left(\frac{1}{y_1} - 1\right)\right\} \tag{120}$$

$$g(y) = \frac{1}{a}\left\{\frac{1}{y_1} - 1\right\} \tag{121}$$

where $y_1 = 1 - ay$. These functions, together with the function $F_1(y)$ determined from equation (77), are evaluated for $a = 1/2$ in table 2(a) and for $a = 5/6$ in table 2(b).

The integrals contained in equation (92) are evaluated by the numerical method employed in the preceding

case and equation (92) gives, for $a = 1/2$,

$$\left.\begin{aligned} \beta_1^{\,2} &= 2.919 \\ \beta_1 &= 1.708 \end{aligned}\right\} \qquad (122)$$

and, for $a = 5/6$,

$$\left.\begin{aligned} \beta_1^{\,2} &= 2.007 \\ \beta_1 &= 1.417 \end{aligned}\right\} \qquad (123)$$

From equation (94) there follows, for $a = 1/2$,

$$\bar{\vartheta}_1(1) = 0.9973 \qquad (124)$$

and, for $a = 5/6$,

$$\bar{\vartheta}_1(1) = 0.9877 \qquad (125)$$

Since equations (124) and (125) indicate a satisfactory agreement between the functions ϑ_1 and $\bar{\vartheta}_1$ in both cases, it is concluded that the functions ϑ_1 and F_1 afford reasonable approximations to the critical deflection and lift modes and that the computed values of β are sufficiently accurate.

If, in the case $a = 1/2$, the initial assumption ϑ_1 is taken to be proportional to the deflection mode ϑ_{sf} predicted by the section-force theory (equation (36)), the value

$$\beta_1 = 1.703 \qquad (126)$$

is obtained and is seen to be in good agreement with the value given in equation (122). However, the check calculation in this case gives the result

$$\bar{\vartheta}_1(1) = 0.9309 \qquad (127)$$

which result, if compared with equation (124), again indicates that the deflection mode ϑ_{sf} does not afford so accurate an approximation to the true deflection mode as does the function defined by equation (76).

The functions ϑ_1 and F_1 are represented graphically in comparison with the function $\vartheta_{sf} = F_{sf}$ predicted by the section-force theory, for the cases $a = 1/2$ and $a = 5/6$, in figures 2(a) and 2(b). If equations (122) and (123) are compared with equations (36) and (37), it is seen that the aerodynamic span effect is responsible for increases of about 32 and 39 percent in the values of the torsional-divergence velocity for the cases $a = 1/2$ and $a = 5/6$, respectively.

3. Linear chord, quadratically decreasing stiffness $\left[c^* = e^* = 1 - ay,\ I^* = (1 - ay)^2\right]$.- The functions H and g in this case are identical with the corresponding functions in the preceding case and are given in equations (120) and (121). From equation (76) the function ϑ_1 is determined in the form

$$\vartheta_1(y) = \frac{1}{a^2}\left\{(1 - y_1) - (1 - a)^2\left(\frac{1}{y_1} - 1\right)\right\} \tag{128}$$

where $y_1 = 1 - ay$. The values of the functions e^*, H, g, and ϑ_1, as well as the values of F_1 determined from equation (77), are listed at the nine points needed for the approximate integration in table 3 for the case $a = 1/2$. Equation (92) then determines the first approximation to β,

$$\left.\begin{aligned} \beta_1^2 &= 5.637 \\ \beta_1 &= 2.374 \end{aligned}\right\} \tag{129}$$

while equation (94) gives

$$\overline{\vartheta}_1(1) = 0.9953 \tag{130}$$

Equation (130) indicates that the results of the first approximation are sufficiently accurate to justify terminating the process.

A comparison of equations (42) and (129) shows that the lifting-line theory predicts a divergence velocity in this case which is about 17 percent higher than the velocity predicted by the section-force theory. The deflection and lift modes according to the two theories are compared in figure 3.

4. Linear chord, quartically decreasing stiffness $\left[c^* = e^* = 1 - ay,\ I^* = (1 - ay)^4\right]$.- The functions ϕ_1, H, and g are determined from equations (76), (91), and (87) as follows:

$$\phi_1(y) = \frac{1 - a}{a^2(3 - a)} \left\{3\left(\frac{1}{y_1} - 1\right) - (1 - a)^2\left(\frac{1}{y_1^3} - 1\right)\right\} \qquad (131)$$

$$H(y) = \frac{1}{6a^2}\left\{(1 + 2a)\left(\frac{1}{y_1^3} - 1\right) - 3\left(\frac{1 - y_1}{y_1^3}\right)\right\} \qquad (132)$$

$$g(y) = \frac{1}{3a}\left\{\frac{1}{y_1^3} - 1\right\} \qquad (133)$$

where $y_1 = 1 - ay$. The functions needed for the computation of β are evaluated in table 4 for the case $a = 1/2$. From equation (92) the approximation

$$\left.\begin{aligned} \beta_1^2 &= 3.908 \\ \beta_1 &= 1.976 \end{aligned}\right\} \qquad (134)$$

is obtained, while the check calculation of equation (94) gives

$$\overline{\phi}_1(1) = 0.9997 \qquad (135)$$

and thus permits a termination of the iterative process.

The lift and deflection modes ϑ_1 and F_1 are compared with the corresponding results given by the section-force theory in figure 4. A comparison of equations (134) and (47) shows that the aerodynamic span effect is responsible for an increase of 20 percent in the value of the torsional-divergence velocity.

For the purpose of further verifying the accuracy of the present procedure the same case has been analyzed by two other methods. First, if the initially assumed deflection mode is taken as a suitable multiple of the mode predicted by the section-force theory (equation (47)) the approximation to β is obtained as

$$\left.\begin{aligned} \beta_1^2 &= 3.898 \\ \beta_1 &= 1.974 \end{aligned}\right\} \tag{136}$$

while equation (94) gives

$$\overline{\vartheta}_1(1) = 0.9332 \tag{137}$$

Second, if the assumed deflection mode is taken to be the deflection corresponding to a uniform distribution of twisting moment along the span (e F = const.) rather than the deflection corresponding to a uniform distribution of lift (F = const.) it is found that

$$\left.\begin{aligned} \beta_1^2 &= 3.907 \\ \beta_1 &= 1.976 \end{aligned}\right\} \tag{138}$$

and

$$\overline{\vartheta}_1(1) = 0.9567 \tag{139}$$

A comparison of these results with equations (134) and (135) indicates that the deflection corresponding to a uniform lift distribution is in closer agreement with the actual divergence mode than is the deflection corresponding to the other assumptions. The remarkable agreement between the computed values of β shows again that insofar as the determination of β is concerned the present procedure is not extremely sensitive to the choice of the initially assumed deflection mode.

5. Linear chord $\left(c^* = e^* = 1 - \frac{1}{2} y\right)$; discontinuous section stiffness (fig. 5). As a final application of the methods of this paper a symmetrical wing is treated in which the chord tapers linearly to a tip value of one-half the root value and the section stiffness varies discontinuously, as indicated in figure 5. The evaluation at nine points of the functions H and g defined in equations (91) and (87) was accomplished by plotting the functions $\frac{1}{I^*(y)}$ and $\frac{1-y}{I^*(y)}$ to a large scale on cross-section paper and in each case counting the squares contained between the corresponding curve and the ordinates at $y = 0$ and $y = y_k$, where $y_k = k/8$, $k = 0,1,\ldots,8$.

If equation (76), which here takes the form

$$\frac{d}{dy}\left[I^*(y)\frac{d\vartheta_1}{dy}\right] = (\text{const.}) \times \left(1 - \frac{1}{2}y\right) \tag{140}$$

is integrated twice and the conditions $\vartheta_1(0) = \vartheta_1'(1) = 0$ are imposed, the function ϑ_1 can be written in the form

$$\vartheta_1(y) = (\text{const.}) \times \int_0^y \frac{\frac{3}{4} - \eta + \frac{1}{4}\eta^2}{I^*(\eta)} d\eta \tag{141}$$

Thus the function ϑ_1 can be conveniently evaluated from the function $\left\{\frac{\frac{3}{4} - y + \frac{1}{4}y^2}{I^*(y)}\right\}$ by graphical integration. The constant is determined, as before, so that $\vartheta_1(1) = 1$.

The values of the functions e^*, H, g, and ϑ_1 as well as the values of the function F_1 determined from equation (77) by the procedure of reference 5, are presented in table 5. If equation (92) is evaluated by the method of approximate integration used in the preceding examples, it is found that

$$\left.\begin{aligned}\beta_1 &= 1.932\\ \bar{\beta}_1 &= 1.390\end{aligned}\right\} \tag{142}$$

Equation (94) then gives

$$\bar{\vartheta}_1(1) = 0.9979 \tag{143}$$

so that, while extreme accuracy should probably not be expected as regards the lift and deflection modes in the neighborhood of the discontinuities, a satisfactory agreement between the functions ϑ_1 and $\bar{\vartheta}_1$ is indicated.

According to the section-force theory the first approximation to the lift function $F(y)$ is given by

$$F_{sf}(y) \approx \left(1 - \frac{1}{2}y\right)\vartheta_1(y) \tag{144}$$

in which case equation (92) gives, as a first approximation,

$$\left.\begin{aligned} \beta_{sf}^2 &= 1.364 \\ \beta_{sf} &= 1.168 \end{aligned}\right\} \tag{145}$$

Comparison of equations (142) and (145) shows that the aerodynamic span effect is responsible in this case for an increase of about 19 percent in the predicted value of the divergence velocity. The lift and deflection modes for the two theories are compared in figure 5.

CONCLUSION

The results of this paper indicate that neglect of the aerodynamic span effect may lead to an appreciable underestimation of the torsional-divergence velocity, the difference between the values obtained with and without neglect of this effect amounting to 17 to 40 percent in the numerical examples presented.

In view of the fact that the cases evaluated concern only wings with a span of about six times the root chord, it seems desirable to consider a greater variety of wings and, in particular, to investigate the relationship between the relative magnitude of the aerodynamic span effect and the magnitude of the aspect ratio.

An extension of the present procedure to the analysis of antisymmetrical deflection modes, as well as to the analysis of the elastic deformation of wings before torsional divergence occurs, can be accomplished without essential difficulty.

Massachusetts Institute of Technology,
Cambridge, Mass., Feb. 1943.

APPENDIX

A FORMULA FOR APPROXIMATE INTEGRATION

If the approximate value of the integral

$$\int_0^1 f(x)\,dx \tag{1}$$

is required, and if the function $f(x)$ is of the form

$$f(x) = p(x)\sqrt{1 - x^2} \tag{2}$$

where $p(x)$ is finite at $x = 1$, conventional formulas such as Simpson's rule fail to give accurate results due to the fact that $f(x)$ has an infinite slope at $x = 1$. A modification of Simpson's rule which takes this fact into account is here derived for a nine-point weighting system.

With the notation $f_k = f(k/8)$, Simpson's rule gives for the range $0 \leq x \leq 3/4$

$$\int_0^{3/4} f(x)\,dx \approx \frac{1}{24}\left\{f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\right\} \tag{3}$$

If, in the range $3/4 \leq x \leq 1$, the function $f(x)$ is approximated by the expression

$$f(x) \approx a_1 \sqrt{1 - x} + a_2 (1 - x) + a_3 (1 - x)^2 \qquad (4)$$

the constants a_1, a_2, and a_3 can be determined so that equation (4) is a true equality at the points $x = 3/4$ and $x = 7/8$ and so that the derivative of the difference between the two sides of that equation is finite at $x = 1$. It then follows that

$$\left.\begin{aligned} a_1 &= \sqrt{2}\, p(1) \\ a_2 &= -4f_6 + 16f_7 - (8 - 2\sqrt{2})\, p(1) \\ a_3 &= 32f_6 - 64f_7 + (32 - 16\sqrt{2})\, p(1) \end{aligned}\right\} \qquad (5)$$

With the approximation of equation (4) there follows

$$\int_{3/4}^{1} f(x)\,dx \approx \frac{1}{12} a_1 + \frac{1}{32} a_2 + \frac{1}{192} a_3 \qquad (6)$$

or

$$\int_{3/4}^{1} f(x)\,dx \approx \frac{1}{24}\left\{f_6 + 4f_7 + \left(\frac{3}{2}\sqrt{2} - 2\right) p(1)\right\} \qquad (7)$$

Equations (3) and (7) can then be combined to give

$$\int_0^1 f(x)\,dx \approx \sum_{k=0}^{8} \sigma_k f(x_k) + \frac{1}{48}(3\sqrt{2} - 4)\left[\frac{f(x)}{\sqrt{1 - x^2}}\right]_{x=1} \qquad (8)$$

where $x_k = k/8$ and σ_k is the weighting coefficient associated with the point x_k by Simpson's rule. The last term in equation (8) is a correction term which takes into account the fact that $f(x)$ has an infinite derivative at the point $x = 1$.

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TABLE 1.- DATA FOR UNIFORM WING $\left(\frac{b}{c} = m\right)$

(a) First Approximation

y	H	g	ϑ_1	F_1	$\bar{\vartheta}_1$
0	0	0	0	0.2680	0
.125	.1172	.1250	.2344	.3040	.2228
.250	.2188	.2500	.4375	.3848	.4263
.375	.3047	.3750	.6094	.4779	.6054
.500	.3750	.5000	.7500	.5518	.7545
.625	.4297	.6250	.8694	.5857	.8691
.750	.4688	.7500	.9375	.5760	.9471
.875	.4922	.8750	.9844	.5157	.9889
1.000	.5000	1.0000	1.0000	0	.9994
			$B + \Sigma A_{2n} = 1.4773$		

(b) Second Approximation

y	ϑ_2	F_2	$\bar{\vartheta}_2$
0	0	0.2624	0
.125	.2229	.2984	.2224
.250	.4265	.3806	.4258
.375	.6057	.4764	.6053
.500	.7549	.5533	.7549
.625	.8696	.5895	.8699
.750	.9476	.5803	.9481
.875	.9895	.5185	.9901
1.000	1.0000	0	1.0006
	$B + \Sigma A_{2n} = 1.4802$		

(c) First Approximation with $\vartheta_1 = \vartheta_{sf}$

y	ϑ_1	F_1	$\bar{\vartheta}_1$	ϑ_2
0	0	0.2399	0	0
.125	.1951	.2730	.2109	.2201
.250	.3827	.3507	.4044	.4220
.375	.5556	.4435	.5758	.6009
.500	.7071	.5211	.7194	.7508
.625	.8315	.5622	.8306	.8668
.750	.9239	.5614	.9067	.9463
.875	.9808	.5083	.9478	.9892
1.000	1.0000	0	.9582	1.0000
	$B + \Sigma A_{2n} = 1.4606$			

TABLE 2.- DATA FOR RECTANGULAR WING WITH QUADRATICALLY TAPERING STIFFNESS $\left(\frac{b}{c} = m\right)$

(a) $I = I_R\left(1 - \frac{1}{2}y\right)^2$

y	H	g	ϑ_1	F_1	$\vartheta_{sf} = F_{sf}$
0	0	0	0	0.2090	0
.125	.1248	.1333	.1616	.2375	.1217
.250	.2484	.2857	.3215	.3069	.2569
.375	.3690	.4615	.4776	.3925	.4032
.500	.4841	.6667	.6265	.4685	.5564
.625	.5897	.9091	.7632	.5163	.7089
.750	.6800	1.2000	.8801	.5290	.8484
.875	.7459	1.5556	.9655	.4910	.9554
1.000	.7726	2.0000	1.0000	0	1.0000
			$B + \Sigma A_{2n} = 1.4287$		

(b) $I = I_R\left(1 - \frac{5}{6}y\right)^2$

y	H	g	ϑ_1	F_1	$\vartheta_{sf} = F_{sf}$
0	0	0	0	0.1425	0
.125	.1305	.1395	.0946	.1615	.0610
.250	.2732	.3158	.1980	.2110	.1293
.375	.4305	.5455	.3119	.2758	.2165
.500	.6047	.8571	.4382	.3428	.3307
.625	.7986	1.3043	.5786	.3974	.4720
.750	1.0124	2.0000	.7335	.4353	.6464
.875	1.2348	3.2308	.8947	.4342	.8541
1.000	1.3801	6.0000	1.0000	0	1.0000
			$B + \Sigma A_{2n} = 1.3327$		

TABLE 3.- DATA FOR LINEARLY TAPERING WING WITH

QUADRATICALLY TAPERING STIFFNESS

$$\left[c = c_R\left(1 - \frac{1}{2}y\right),\ I = I_R\left(1 - \frac{1}{2}y\right)^2,\ \frac{b^2}{S} = \frac{4}{3}m\right]$$

y	e^*	H	g	ϑ_1	F_1	ϑ_{sf}	F_{sf}
0	1.0000	0	0	0	0.1868	0	0
.125	.9375	.1248	.1333	.1833	.2145	.1492	.1399
.250	.8750	.2484	.2857	.3571	.2783	.3094	.2708
.375	.8125	.3690	.4615	.5192	.3490	.4731	.3844
.500	.7500	.4841	.6667	.6667	.3980	.6312	.4734
.625	.6875	.5897	.9091	.7955	.4109	.7740	.5321
.750	.6250	.6800	1.2000	.9000	.3973	.8908	.5568
.875	.5625	.7459	1.5556	.9722	.3715	.9704	.5458
1.000	.5000	.7726	2.0000	1.0000	0	1.0000	.5000
				$B+\Sigma A_{2n} = 1.2058$			

TABLE 4.- DATA FOR LINEARLY TAPERING WING WITH

QUARTICALLY TAPERING STIFFNESS

$$\left[c = c_R\left(1 - \frac{1}{2}y\right),\ I = I_R\left(1 - \frac{1}{2}y\right)^4,\ \frac{b^2}{S} = \frac{4}{3}m\right]$$

y	e^*	H	g	ϑ_1	F_1	ϑ_{sf}	F_{sf}
0	1.0000	0	0	0	0.1409	0	0
.125	.9375	.1333	.1424	.1173	.1567	.0826	.0774
.250	.8750	.2838	.3285	.2443	.2056	.1858	.1626
.375	.8125	.4533	.5762	.3810	.2691	.3066	.2491
.500	.7500	.6420	.9136	.5259	.3217	.4579	.3434
.625	.6875	.8465	1.3849	.6754	.3485	.6214	.4272
.750	.6250	1.0560	2.0640	.8208	.3550	.7891	.4932
.875	.5625	1.2419	3.0791	.9429	.3491	.9332	.5249
1.000	.5000	1.3333	4.6667	1.0000	0	1.0000	.5000
				$B+\Sigma A_{2n} = 1.1686$			

TABLE 5.- DATA FOR LINEARLY TAPERING WING WITH

DISCONTINUOUS STIFFNESS VARIATION

$$\left[c = c_R\left(1 - \frac{1}{2}y\right),\ \frac{b^2}{S} = \frac{4}{3}m\right]$$

y	e*	H	g	ϑ_1	F_1	$c^*\vartheta_1$
0	1.0000	0	0	0	0.1455	0
.125	.9375	.1172	.1250	.0656	.1582	.0615
.250	.8750	.2188	.2500	.1200	.2364	.1050
.375	.8125	1.0781	1.5000	.5606	.3523	.4555
.500	.7500	1.7813	2.7500	.9043	.4463	.6782
.625	.6875	1.8435	2.8927	.9333	.4731	.6417
.750	.6250	1.9041	3.0891	.9599	.4352	.5999
.875	.5625	1.9616	3.4061	.9836	.3770	.5533
1.000	.5000	2.0038	4.3267	1.0000	0	.5000
				$B + \Sigma A_{2n} = 1.2400$		

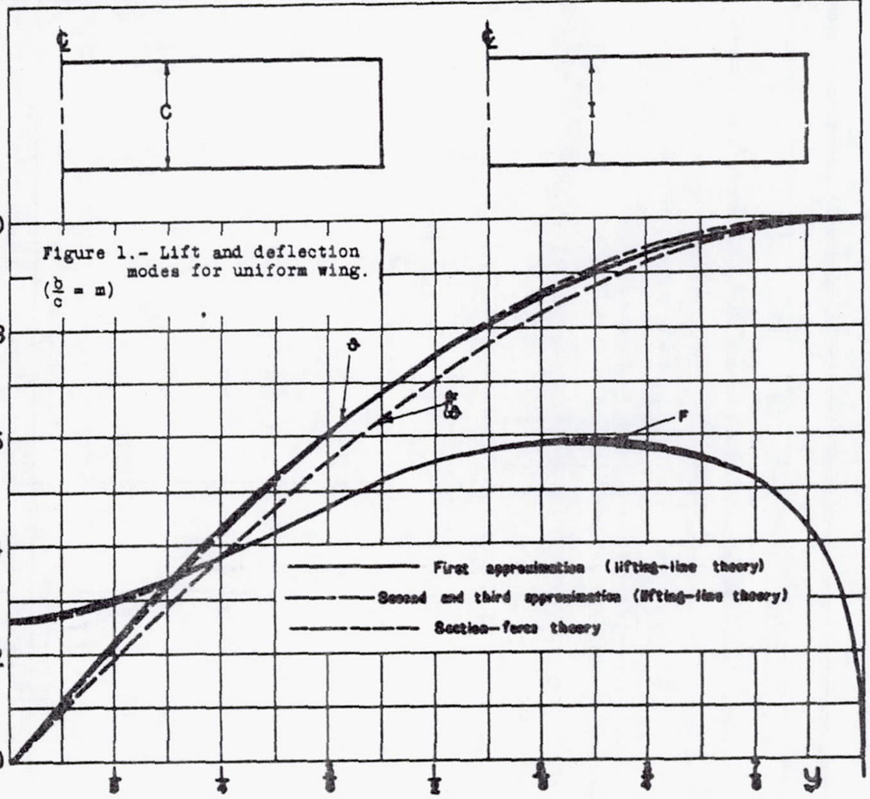

Figure 1.- Lift and deflection modes for uniform wing. $(\frac{b}{c} = m)$

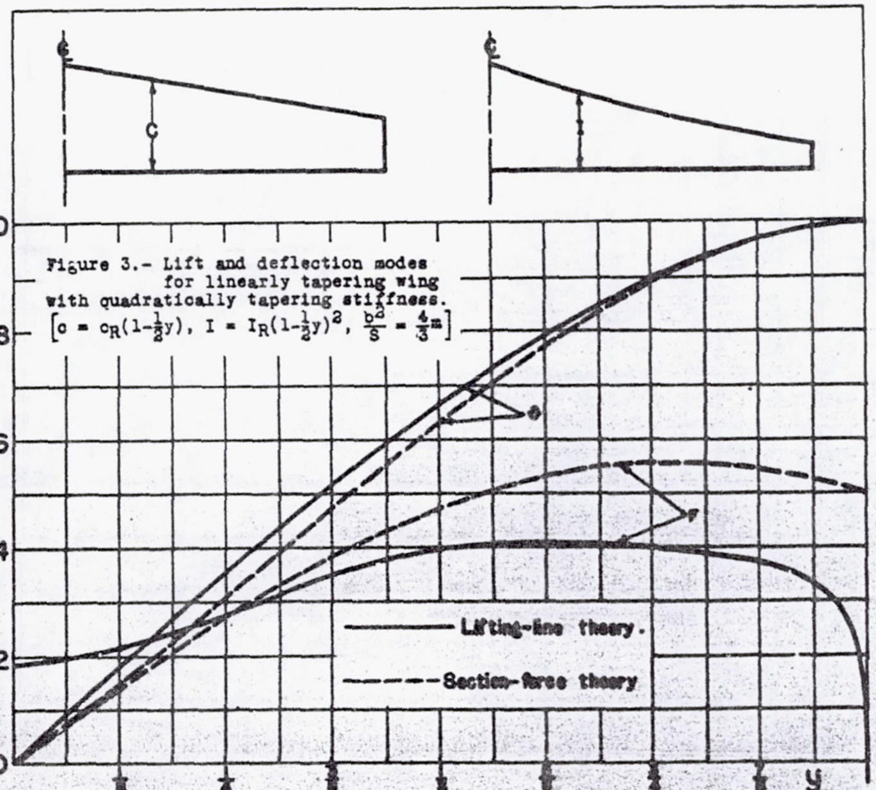

Figure 3.- Lift and deflection modes for linearly tapering wing with quadratically tapering stiffness. $\left[c = c_R(1-\frac{1}{2}y),\ I = I_R(1-\frac{1}{2}y)^2,\ \frac{b^2}{S} = \frac{4}{3}m\right]$

(1 block = 10 divisions on 1/32" Arch. scale)

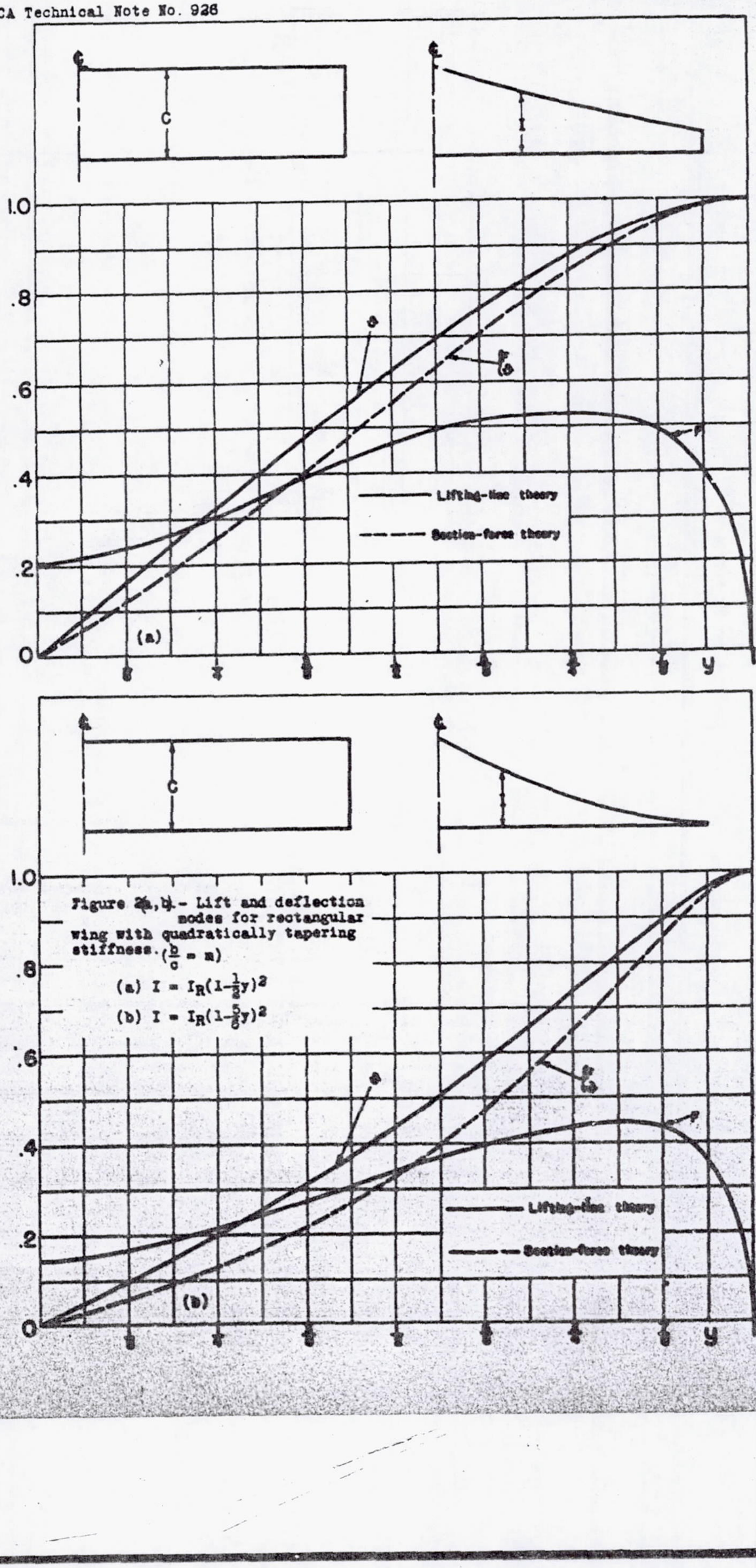

Figure 2(a,b).- Lift and deflection modes for rectangular wing with quadratically tapering stiffness. ($\frac{b}{c}$ = m)

(a) $I = I_R(1-\frac{1}{2}y)^2$

(b) $I = I_R(1-\frac{5}{6}y)^2$

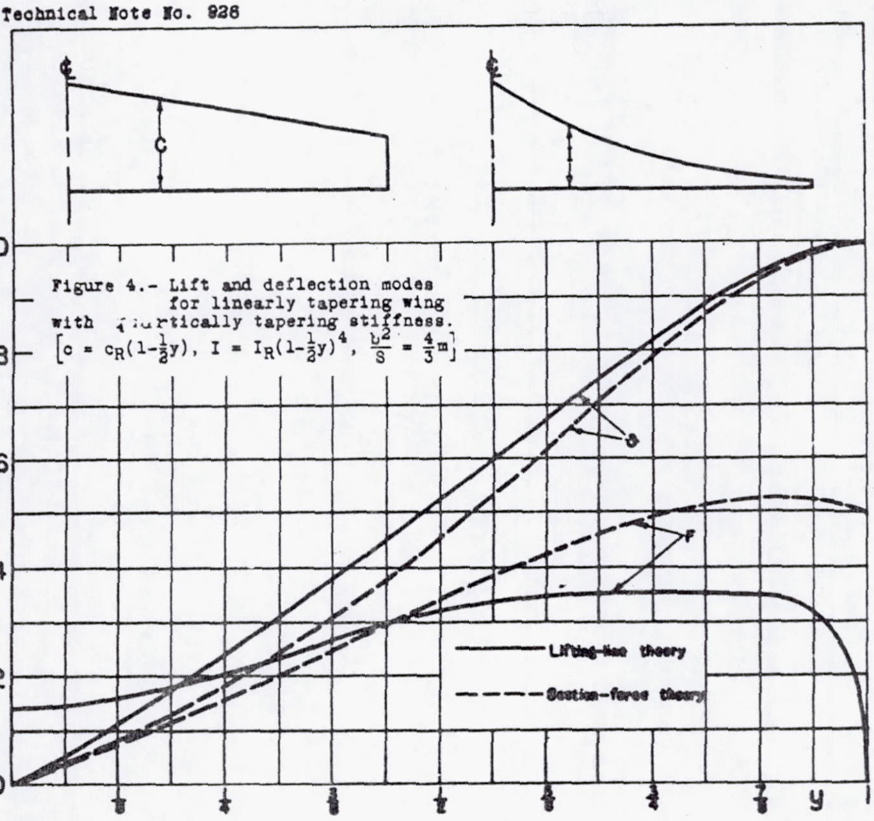

Figure 4.- Lift and deflection modes for linearly tapering wing with quartically tapering stiffness. $\left[c = c_R(1-\frac{1}{2}y),\ I = I_R(1-\frac{1}{2}y)^4,\ \frac{b^2}{S} = \frac{4}{3}m\right]$

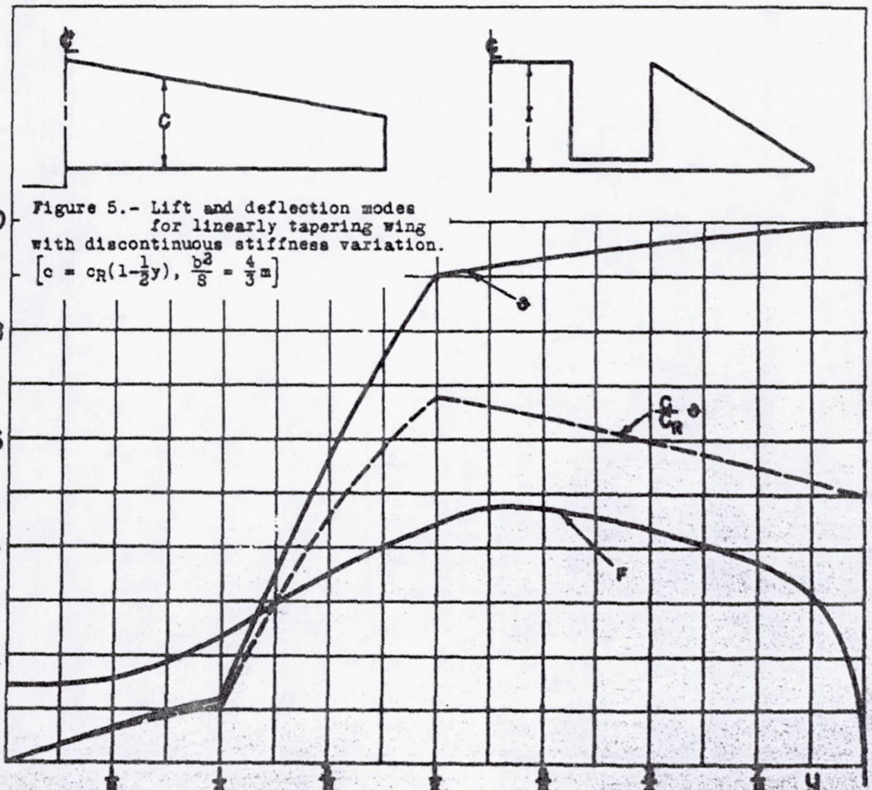

Figure 5.- Lift and deflection modes for linearly tapering wing with discontinuous stiffness variation. $\left[c = c_R(1-\frac{1}{2}y),\ \frac{b^2}{S} = \frac{4}{3}m\right]$

(1 block = 10/32")

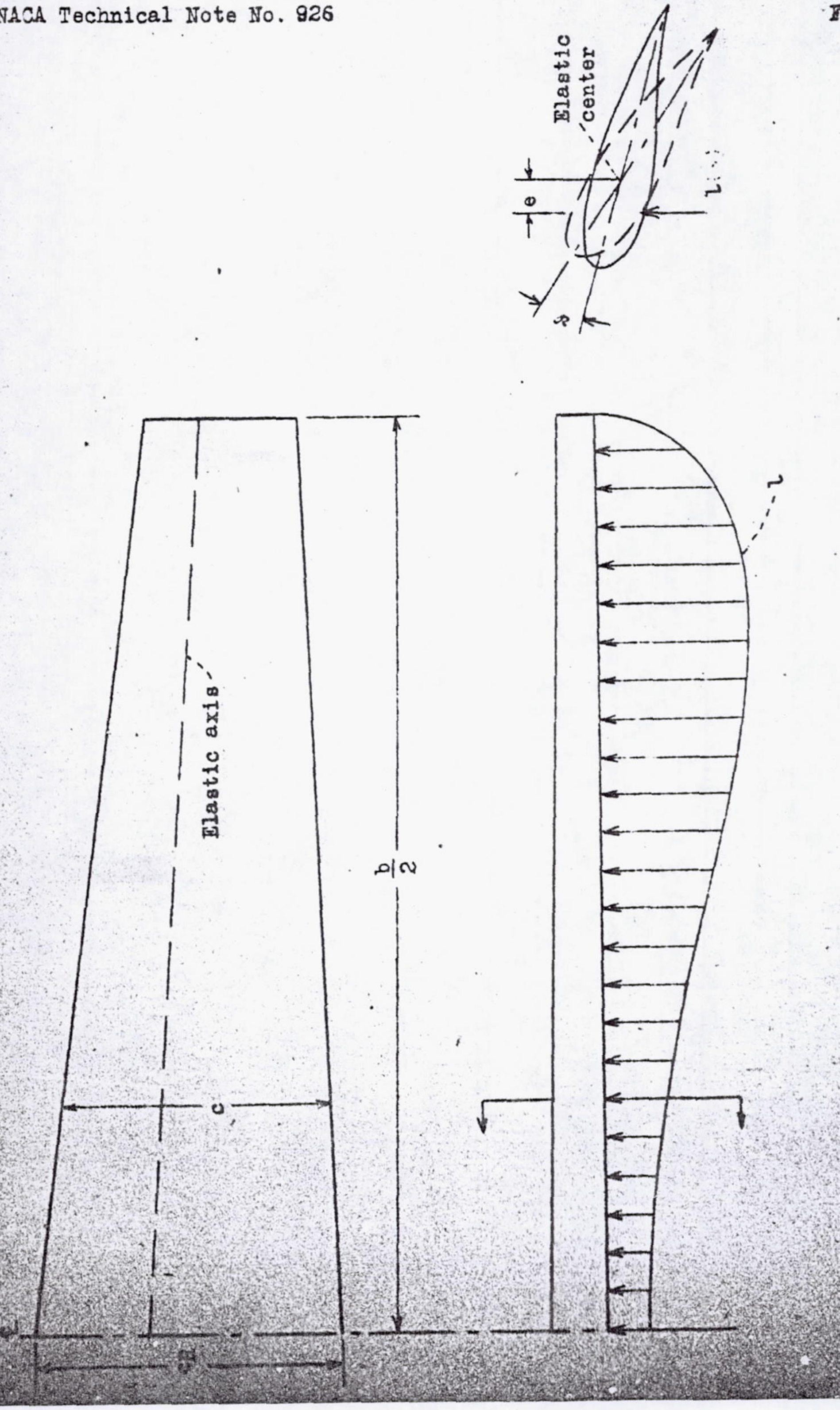

Figure 6.- Sketch of wing, showing lift distribution and elastic deformation.